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Precession of Orbiting Gyroscope in Higher-Order Gravitational Field Caused by Rotating Body

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The metric for a spinning massive object with any shape and composition is found by the use of linearized higher-order theory of gravitation. The geodesic and the Lense–Thirring precessions for an orbiting gyroscope in a general weak higher-order gravitational field are considered. The influences of the additional Yukawa forces included in the linearized higher-order gravitation on the precessions are investigated.

KEY WORDS: higher-order gravity; geodetic precession; Lense–Thirring precession.

1. INTRODUCTION

The precession for an orbiting gyroscope is an important phenomenon in astrophysics, which may be used to test the predictions of general relativity and other gravitational theories. Einstein's general theory of relativity predicts that a gyroscope circling the Earth in a low circular polar orbit with altitude 650 km will precess about 6.6 arcsec/year in the orbital plane (geodetic precession) and about 42 milliarcsec/year perpendicular to the orbital plane (Lense–Thirring precession) (Misner *et al.*, 1973; Ohanian and Ruffini, 1994; Will, 1993).

Since the times of Eddington and Weyl, higher-order gravitational theories have been discussed by several generations of scientists (Havas, 1977), and applied to quantum gravity (Stelle, 1997), early cosmology (Barrow and Ottewill, 1983), pure gravitational inflationary model for the universe (Mijic *et al.*, 1986), eliminating the singularities in gravity (Treder, 1975), explaining the dark matter in the universe (Mannheim and Kazanas, 1989, 1994), and so on. Eddington (1924) and Weyl (1952) pointed out that higher-order theories of gravitation were observationally equivalent to Einstein's because they included as one of their solutions the (exterior) Schwarzchild metric. It was noted by Pauli (1921) and Buchdahl (1948)

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that every vacuum solution (including the Schwarzchild solution) of general relativity is also a solution of any fourth-order theory.

In this paper, we consider the movement of an orbiting gyroscope in linearized higher-order gravitational field due to a rotating object. We first find the general solution of linearized field equation for higher-order gravitational theory, and the metric for a spinning body with any shape and composition. Next, we derive the precession equations for an orbiting gyroscope, give the geodesic and the Lense–Thirring precessions for an orbiting gyroscope, and discuss the influences of the additional forces of Yukawa type in higher-order gravity on the precessions. Finally, we investigate the precessions of an orbiting gyroscope in Earth's higher-order gravitational field and in higher-order gravitational field due to a neutron star, and compare the precessions predicted by higher-order theory to that predicted by general relativity.

2. GENERAL SOLUTION FOR THE LINEARIZED HIGHER-DERIVATIVE FIELD EQUATIONS OF GRAVITY

A general action for higher-order gravitation may be written as

$$S = \int d^4x[\sqrt{-g}(R + aR^2 + bR_{\mu\nu}R^{\mu\nu}) - kL_m] \tag{2.1}$$

where k is Einstein's constant with $k = 8\pi G/c^4$, a and b are two new parameters, L_m is the matter Lagrangian, $R_{\mu\nu}$ is the Ricci tensor, and $R = g^{\mu\nu}R_{\mu\nu}$.

The variation of the action (2.1) with respect to the metric $g_{\mu\nu}$ yields the higher-derivative field equations

$$G_{\mu\nu} = G^{(E)}_{\mu\nu} + aG^{(1)}_{\mu\nu} - bG^{(2)}_{\mu\nu} = kT_{\mu\nu} \tag{2.2}$$

where

$$\begin{aligned}
G^{(E)}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \\
G^{(1)}_{\mu\nu} &= 2R_{;\mu;\nu} - 2g_{\mu\nu}R^{;\sigma}_{;\sigma} + 2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 \\
G^{(2)}_{\mu\nu} &= g_{\mu\nu}R^{\sigma\rho}_{;\sigma;\rho} + g^{\sigma\rho}(R_{\mu\nu;\sigma;\rho} - R_{\mu\sigma;\nu;\rho} - R_{\nu\sigma;\mu;\rho}) \\
&\quad - 2R_{\mu\sigma}R^{\sigma}_{\nu} + \frac{1}{2}g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho}
\end{aligned}$$

$T_{\mu\nu}$ is the energy–momentum tensor of the sources of the gravitational field. A semicolon denotes covariant differentiation. To obtain the linear approximation, i.e., the linear equations in the components of the metric tensor $g_{\mu\nu}$, we first drop the quadratic terms in R and $R_{\mu\nu}$, as well as $2R\,R_{\mu\nu}$ in Eq. (2.2). As a consequence, Eq. (2.2) reduces to

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + 2a\big(R_{;\mu;\nu} - 2g_{\mu\nu}R^{;\sigma}_{;\sigma}\big) - b\big(R^{;\sigma}_{\mu\nu;\sigma} + g_{\mu\nu}R^{\sigma\rho}_{;\sigma;\rho} - R^{;\sigma}_{\mu\sigma;\nu} - R^{;\sigma}_{\nu\sigma;\mu}\big) = kT_{\mu\nu} \tag{2.3}$$

The gravitational field is supposed to be weak. So we put

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.4}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the flat space-time metric and $h_{\mu\nu}$ characterizes the contribution to the metric because of the material fields.

Neglecting terms of order $h^2, h^3, \ldots$, and denoting the d' Alembertian operator $\eta^{\alpha\beta}\partial_\alpha\partial_\beta$ by $\Box$, we rewrite Eq. (2.3) as

$$(1 - b\,\Box)\left(R_{\mu\nu} - \frac{1}{2}\eta_{\eta\nu}R\right) - (2a + b)(\eta_{\mu\nu}\,\Box\,R - R_{,\mu\nu}) = kT_{\mu\nu} \tag{2.5}$$

In these equations as well as in those that follow, $R_{\mu\nu}$ and R must be replaced by their respective first-order expressions.

$$R_{\mu\nu} = -\frac{1}{2}\,\Box\,h_{\mu\nu} + \frac{1}{2}\big(\gamma^{\rho}_{\mu,\nu\rho} + \gamma^{\rho}_{\nu,\mu\rho}\big) \tag{2.6}$$

$$R = -\frac{1}{2}\,\Box\,h + \gamma^{\mu\nu}_{,\mu\nu} \tag{2.7}$$

Here, the comma used as an index denotes partial differentiation, indices are raised (lowered) using $\eta^{\mu\nu}(\eta_{\mu\nu})$, and the quantities $\gamma_{\mu\nu}$ are defined by

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.8}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. Note, however, that we have the identities $R^{\nu}_{\mu,\nu} = \frac{1}{2}R_{,\mu}$ in the case of the linear field approximation, which allows us to conclude that

$$\left[(1 - b\,\Box)\left(R^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R\right) - (2a + b)(\eta^{\mu\nu}\,\Box\,R - R^{,\mu\nu})\right]_{,\nu} = 0 \tag{2.9}$$

Therefore the linearized Eq. (2.5) implies

$$T^{\mu\nu}_{,\nu} = 0 \tag{2.10}$$

Equation (2.10) is identically the energy-momentum laws of the special theory of relativity, which imply the special relativistic equations of motion (which contain no gravitational interactions)

Contracting Eq. (2.5) by $\eta^{\mu\nu}$ and putting $T = \eta^{\alpha\beta}T_{\alpha\beta}$ yield

$$(2a + b)\,\Box\,R = -\frac{1}{3}kT - \frac{1}{3}(1 - b\,\Box)R \tag{2.11}$$

Substituting Eq. (2.11) into (2.5) and taking Eq. (2.6) into account, we can rewrite the linearized field equations in the equivalent form

$$(1 - b\,\Box)\left(\Box\, h_{\mu\nu} + \frac{1}{3}\eta_{\mu\nu}R\right) - (\Gamma_{\mu,\nu} + \Gamma_{\nu,\mu}) = -2k\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right) \quad (2.12)$$

where

$$\Gamma_\mu = (1 - b\,\Box)\gamma^{\nu}_{\mu,\nu} + (2a + b)R_{,\mu} \equiv \Gamma_\mu(h_{\alpha\beta}) \quad (2.13)$$

The Teyssandier gauge is defined by the subsidiary condition $\Gamma_\mu = 0$ on the potentials (Teyssandier, 1989). Let us then show in which way this gauge can be realized initially assuming $\Gamma_\mu \neq 0$. Indeed, let $h_{\mu\nu}$ be a solution of Eq. (2.12). Under an arbitrary infinitesimal coordinate transformation $x^\mu \to \bar{x}^\mu = x^\mu + \Lambda^\mu$, where Λ^μ is an infinitesimal vector field, $h_{\mu\nu}$ transforms into $\bar{h}_{\mu\nu}(x) = h_{\mu\nu} - \Lambda_{\mu,\nu} - \Lambda_{\nu,\mu}$, which is also a solution of Eq. (2.12) since $R(h_{\alpha\beta})$ transforms into $R(\bar{h}_{\alpha\beta}) = R(h_{\alpha\beta})$ and $\Gamma_\mu(h_{\alpha\beta})$ transforms into

$$\bar{\Gamma}_\mu(\bar{h}_{\alpha\beta}) = \Gamma_\mu(h_{\alpha\beta}) - (1 - b\,\Box)\,\Box\,\Lambda_\mu \neq \Gamma_\nu(h_{\alpha\beta})\frac{\partial x^\nu}{\partial \bar{x}^\mu}$$

The Teyssandier gauge for the vanishing of $\bar{\Gamma}_\mu$ can now be realized by demanding that $\Gamma_\mu = (1 - b\,\Box)\,\Box\Lambda_\mu$. Thus, the problem of solving the linearized field equations of higher-derivative gravity is completely equivalent to that of solving the system of equations as follows:

$$(1 - b\,\Box)\left(\Box\, h_{\mu\nu} + \frac{1}{3}\eta_{\mu\nu}R\right) = -2k\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right) \quad (2.14)$$

Now, we define the quantities $\psi_{\mu\nu}$ by

$$\psi_{\mu\nu} = -\frac{1}{\lambda_1^2}\left(\Box\, h_{\mu\nu} + \frac{1}{3}R\eta_{\mu\nu}\right) \quad (2.15)$$

where we have assumed that $\lambda_1^2 = -b^{-1}$. It is easily seen from Eqs. (2.14) and (2.15) that the $\psi_{\mu\nu}$ satisfies the equations

$$(\Box + \lambda_1^2)\psi_{\mu\nu} = 2k\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right) \quad (2.16)$$

Eliminating the term of $\lambda_1^2\psi_{\mu\nu}$ in Eq. (2.16) by using Eq. (2.15) yields

$$\Box\,(h_{\mu\nu} - \psi_{\mu\nu}) + \frac{1}{3}R\eta_{\mu\nu} = -2k\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right) \quad (2.17)$$

Equation (2.11) may be rewritten as

$$(\Box + \lambda_0^2)\phi = \frac{1}{3}kT \quad (2.18)$$

where we have assumed that $\lambda_0^2 = 1/2(3a + b)$ and

$$\phi = \frac{R}{3\lambda_0^2} \tag{2.19}$$

Eliminating the factor $\lambda_0^2\phi$ by combining Eqs. (2.18) with (2.19), we obtain

$$R = -3\,\Box\,\phi - kT \tag{2.20}$$

Substituting Eq. (2.20) into (2.17), we have

$$\Box(h_{\mu\nu} - \psi_{\mu\nu} - \phi\eta_{\mu\nu}) = -2k\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) \tag{2.21}$$

On the other hand, the linearized field equations for Einstein's gravitation in the harmonic gauge $\gamma^{(E)\nu}_{\mu,\nu} = 0$, where $\gamma^{(E)}_{\mu\nu} = h^{(E)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{(E)}$, are

$$\Box\, h^{(E)}_{\mu\nu} = -2k\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) \tag{2.22}$$

Thus, comparing Eq. (2.21) with Eq. (2.22), we may say that if $\lambda_0 \neq 0$ and $\lambda_1 \neq 0$, the general solutions of the linearized field equations for higher-derivative gravity in the Teyssandier gauge are given by

$$h_{\mu\nu} = h^{(E)}_{\mu\nu} + \psi_{\mu\nu} + \phi\eta_{\mu\nu} \tag{2.23}$$

where $h^{(E)}_{\mu\nu}$ is a solution of Eq. (2.22), which describes a massless tensor field; $\psi_{\mu\nu}$ a solution of Eq. (2.16), which describes a massive tensor field; ϕ a solution of Eq. (2.18), which describes a massive scalar field. It is worth mentioning that λ_0 and λ_1 can be real or imaginary according to the signs of b and $3a + b$. In next discussion, we will assume that $\lambda_1^2 > 0$ ($b < 0$) and $\lambda_0^2 > 0$ ($3a + b > 0$), which corresponds to the absence of tachyone (both positive and negative energy) in the dynamical field (Accioly and Azeredo, 2000), to assure asymptotic agreement of the theory with Newton's law.

3. METRIC OF ROTATING BODY

For a time-independent system, or a system which changes so slowly that retardation effects may be ignored, the solutions of Eqs. (2.22), (2.16), and (2.18) are as follows respectively:

$$h^{(E)}_{\mu\nu} = -\frac{k}{2\pi}\int \frac{T_{\mu\nu}(\vec{r}\,') - \eta_{\mu\nu}T(\vec{r}\,')/2}{|\vec{r} - \vec{r}\,'|}\, d^3\vec{r}\,' \tag{3.1}$$

$$\psi_{\mu\nu} = \frac{k}{2\pi}\int \frac{T_{\mu\nu}(\vec{r}\,') - \eta_{\mu\nu}T(\vec{r}\,')/3}{|\vec{r} - \vec{r}\,'|}\, e^{-\lambda_1|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' \tag{3.2}$$

$$\phi = -\frac{k}{12\pi} \int \frac{T(\vec{r}')}{|\vec{r} - \vec{r}'|} \, e^{-\lambda_0|\vec{r}-\vec{r}'|} \, d^3\vec{r}' \tag{3.3}$$

If we suppose the body producing the metric field rotates stably with respect to its center and the pressure is negligible, the energy-momentum tensor reduces to

$$T^{\mu\nu} = c^2 \rho(\vec{r}) u^\mu u^\nu \tag{3.4}$$

implying $T = c^2\rho(\vec{r})$, where u^μ is the four-velocity and $\rho(\vec{r})$ is the time-independent matter density measured in the frame rotating with body.

Introducing the series expansions

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + \frac{1}{r^3} \sum_{i=1}^{3} x^i x'^i + \cdots$$

$$\frac{1}{|\vec{r} - \vec{r}'|} \, e^{-\lambda|\vec{r}-\vec{r}'|} = e^{-\lambda r} \left(\frac{1}{r} + \frac{1 + \lambda r}{r^3} \sum_{i=1}^{3} x^i x'^i + \cdots \right)$$

and the energy-momentum tensor from Eq. (3.4) into (3.1), (3.2), and (3.3), choosing the mass center of the body as the origin of coordinates, assuming that the distance r to a field point from the origin of coordinates is larger than the distance r' to a point in the source from that, and then calculating the integrations, we have, in the case of the dipole approximation

$$h_{00}^{(E)} = h_{ii}^{(E)} = -\frac{2GM}{c^2 r}, \quad h_{i0}^{(E)} = \frac{2G}{c^3 r^3} (\vec{r} \times \vec{J})_i \tag{3.5}$$

$$\psi_{00} = 2\psi_{ii} = \frac{8GM}{3c^2 r} \, e^{-\lambda_1 r}, \quad \psi_{i0} = -\frac{2G}{c^3 r^3} (1 + \lambda_1 r)(\vec{r} \times \vec{J})_i \, e^{-\lambda_1 r} \tag{3.6}$$

$$\phi = -\frac{2GM}{3c^2 r} \, e^{-\lambda_0 r} \tag{3.7}$$

where the Latin indices run from 1 to 3 and $\vec{J}$ is the angular momentum of the rotating system defined by

$$\vec{J} = \int \rho(\vec{r}')(\vec{r}' \times \vec{V}) d^3\vec{r}' \tag{3.8}$$

We substitute Eqs. (3.5)–(3.7) into (2.23), obtaining

$$h_{00} = \frac{2}{c^2}\Phi, \quad h_{i0} = \frac{1}{c^3} A_i, \quad h_{ij} = \frac{2}{c^2} \chi \eta_{ij} \tag{3.9}$$

where

$$\Phi = -\frac{GM}{r} \left(1 + \frac{1}{3} \, e^{-\lambda_0 r} - \frac{4}{3} \, e^{-\lambda_1 r} \right) \tag{3.10}$$

$$A_i = \frac{2G}{r^3}[1 - (1 + \lambda_1 r)e^{-\lambda_1 r}](\vec{r} \times \vec{J})_i \tag{3.11}$$

$$\chi = \frac{GM}{r}\left(1 - \frac{1}{3}\, e^{-\lambda_0 r} - \frac{2}{3}\, e^{-\lambda_1 r}\right) \tag{3.12}$$

Here Φ and χ are two metric potentials, which are same as the results found by Teyssandier (1989) and Accioly (2000) in the case of the matter system being stationary. Φ denotes the modified Newton's gravitational potential (the gravito-electric field) and $\vec{A} = (A_1, A_2, A_3)$ is referred to as a gravitational vector potential (the gravito–magnetic field) in the framework of linearized higher-order theory of gravitation.

In summary, we may write the (Lense–Thirring) line element as

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2\, dt^2 + \frac{2}{c^2}\vec{A}\cdot d\vec{r}\; dt - \left(1 + \frac{2\chi}{c^2}\right) d\vec{r}^{\,2} \tag{3.13}$$

which is valid to first order in field intensity and source velocity, and will serve as a basis for calculating the gyroscope precession.

4. ORBITING GYROSCOPE PRECESSION

An orbiting gyroscope has its spin axis paralled-displaced in accord with the metric (3.13). The paralled-displacement equation for the gyro spin S^α is

$$\frac{dS^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} S^\mu \frac{dx^\nu}{ds} = 0 \tag{4.1}$$

where $\Gamma^\alpha_{\mu\nu}$, the Christoffel symbol of the second kind, is defined by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) \tag{4.2}$$

The space components of Eq. (4.1) are

$$\frac{dS^i}{ds} + \Gamma^i_{\mu\nu} S^\mu \frac{dx^\nu}{ds} = 0 \tag{4.3}$$

Substituting the metric given from Eq. (3.13) into (4.2), to lowest order in the potentials the Christoffel symbols are

$$\begin{aligned} &\Gamma^0_{0i} = \frac{1}{c^2}\Phi_{,i}, \quad \Gamma^i_{00} = \frac{1}{c^2}\Phi_{,i}, \quad \Gamma^i_{j0} = \frac{1}{2c^3}(A_{j,i} - A_{i,j}),\\ &\Gamma^i_{jk} = \frac{1}{c^2}(\chi_{,i}\eta_{jk} - \chi_{,j}\eta_{ik} - \chi_{,k}\eta_{ij}) \end{aligned} \tag{4.4}$$

Since the gyro spin four-vector is perpendicular to the velocity four-vector, i.e., $S_\mu u^\mu = 0$, which is equivalent to that the gyro spin has no zero component in its rest frame (Adler and Silbergleit, 2000), the zero component in another frame is easily obtained, and to first order in the satellite velocity $\vec{\upsilon}$ it is given by $S^0 = \vec{S} \cdot \vec{\upsilon}/c$. Substitution of the Christoffel symbols given by Eq. (4.4) into the spin equation of motion (4.3) gives

$$\frac{dS^i}{dt} + \frac{1}{c^2}[(\vec{S} \cdot \vec{\upsilon})\Phi_{,i} + (\vec{\upsilon} \cdot \nabla\chi)S^i + (S \cdot \nabla\chi)\upsilon^i - (\vec{S} \cdot \vec{\upsilon})\chi_{,i}] + \frac{1}{2c^2}(A_{j,i} - A_{i,j})S^j = 0 \tag{4.5}$$

Breaking the drift rate into two parts, the geodetic drift rate due to the metric potential Φ and χ, and the Lense–Thirring precession rate due to the vector potential $\vec{A}$, and then separating also symmetric and antisymmetric parts of the geodesic effect, we arrive at, in a three-dimensional vector notation

$$\dot{\vec{S}} = \dot{\vec{S}}_G + \dot{\vec{S}}_{LT} \tag{4.6}$$

with

$$\dot{\vec{S}}_{LT} = \vec{\Omega}_{LT} \times \vec{S}, \quad \vec{\Omega}_{LT} = \frac{1}{2c^2}\nabla \times \vec{A} \tag{4.7}$$

and

$$\dot{\vec{S}}_G = \vec{\Omega}_G \times \vec{S} + \frac{1}{2c^2}[(\vec{S} \cdot \vec{\upsilon})\nabla\Phi + (\vec{S} \cdot \nabla\Phi)\vec{\upsilon}] + \frac{1}{c^2}(\vec{\upsilon} \cdot \nabla\chi)\vec{S} \tag{4.8}$$

$$\vec{\Omega}_G = \frac{1}{2c^2}[\nabla(\Phi - 2\chi)] \times \vec{\upsilon} \tag{4.9}$$

where $\vec{\Omega}_{LT}$ and $\vec{\Omega}_G$ are the instantaneous values of the Lense–Thirring and the geodesic precessions, respectively. Since $\vec{\Omega}_{LT}$ is the curl of the gravitational vector potential the Lense–Thirring precession rate is the analog of the magnetic field in magnetostatics theory. The effects of the second and third terms to the right of Eq. (4.8) almost vanish when averaged over any reasonable satellite orbit. To see this, using Newton's law in the form $\nabla\Phi = -\dot{\vec{\upsilon}}$ and taking account of that $\chi < -\Phi$, we may write

$$\frac{1}{c^2}|\langle\vec{\upsilon} \cdot \nabla\chi\rangle\vec{S}| < \frac{1}{c^2}|-\langle\vec{\upsilon} \cdot \nabla\Phi\rangle\vec{S}| = \frac{1}{c^2}\left|\left\langle\vec{\upsilon} \cdot \frac{d\vec{\upsilon}}{dt}\right\rangle\vec{S}\right| = \frac{1}{2c^2}\left|\left\langle\frac{d\vec{\upsilon}^2}{dt}\right\rangle\vec{S}\right| = \left|\frac{\nabla\vec{\upsilon}^2}{2c^2T}\vec{S}\right| \tag{4.10}$$

where $\Delta\vec{\upsilon}^2$ is the change in the velocity squared in total time T, and it is assumed that the drift rate is small. In reality, if the orbit is periodic, this quantity will be zero, and for a nearly periodic orbit it will be very small.

In a similar way to above analysis, we have

$$\frac{1}{2c^2}|\langle(\vec{S}\cdot\vec{\upsilon})\nabla\Phi + (S\cdot\nabla\Phi)\vec{\upsilon}\rangle_j| = \left|-\frac{\Delta(\upsilon^i\upsilon^j)}{2c^2T}\vec{S}_i\right| \tag{4.11}$$

This quantity is zero for a periodic orbit and very small for a nearly periodic orbit.

In summary, the average precession rate of the gyro spin is

$$\begin{aligned}\langle\dot{\vec{S}}\rangle &= \langle\dot{\vec{S}}_G\rangle + \langle\dot{\vec{S}}_{LT}\rangle, \quad \langle\dot{\vec{S}}_G\rangle = \langle\vec{\Omega}_G\rangle\times\vec{S},\\ \langle\dot{\vec{S}}_{LT}\rangle &= \langle\vec{\Omega}_{LT}\rangle\times\vec{S}\end{aligned} \tag{4.12}$$

where the values of the Lense–Thirring and the geodesic precessions are respectively given by Eqs. (4.7), (4.8), and (4.9).

5. DISCUSSION AND CONCLUSION

We found the metric due to a rotating body, using linearized higher-order field equations of gravitation, and derived the precession equation for an orbiting gyroscope in the rotary higher-order gravitational field. In this section, we discuss the effects of the Yukawa potentials included in linearized higher-order gravitational field on the precessions. Substitution of Eqs. (3.10), (3.11), and (3.12) into (4.7) and (4.9) gives.

$$\vec{\Omega}_{LT} = \vec{\Omega}^{(E)}_{LT} + \vec{\Omega}^{(A)}_{LT}, \quad \vec{\Omega}_G = \vec{\Omega}^{(E)}_G + \vec{\Omega}^{(A)}_G \tag{5.1}$$

with

$$\vec{\Omega}^{(E)}_{LT} = -\frac{G}{c^2r^5}[3(\vec{J}\cdot\vec{r})\vec{r} - r^2\vec{J}] \tag{5.2}$$

$$\vec{\Omega}^{(A)}_{LT} = \frac{G}{c^2r^5}\left[\left(3+3\lambda_1 r+\lambda_1^2r^2\right)(\vec{J}\cdot\vec{r})\vec{r} - \left(1+\lambda_1 r+\lambda_1^2r^2\right)r^2\vec{J}\right]e^{-\lambda_1 r} \tag{5.3}$$

$$\vec{\Omega}^{(E)}_G = \frac{GM}{2c^2r^3}(\vec{r}\times\vec{\upsilon}) \tag{5.4}$$

$$\vec{\Omega}^{(A)}_G = -\frac{GM}{6c^2r^3}[(1+\lambda_0 r)e^{-\lambda_0 r} - 8(1+\lambda_1 r)e^{-\lambda_1 r}](\vec{r}\times\vec{\upsilon}) \tag{5.5}$$

where $\vec{\Omega}^{(E)}_{LT}$ and $\vec{\Omega}^{(E)}_G$ are respectively the Lense–Thirring and the geodesic precessions of an orbiting gyroscope given by general relativity, $\vec{\Omega}^{(A)}_{LT}$ and $\vec{\Omega}^{(A)}_G$ are respectively the Lense–Thirring and the geodesic effects due to the additional forces of the Yukawa type in linearized higher-order gravity.

Suppose a satellite which contains a set of gyroscopes intended to test the predictions of gravitational theories will be circling the Earth on an orbit with altitude $h = 650$ km, which is described by $r = r_0 + h$, where r_0 expresses the

Earth radius. For simplicity, we assume further the orbital plane is perpendicular to $\vec{J}$. Comparing the precession effects of the Yukawa force and general relativity, from Eqs. (5.2) to (5.5) we arrive at

$$\frac{\Omega_{LT}^{(A)}}{\Omega_{LT}^{(E)}} = \frac{\left|\vec{\Omega}_{LT}^{(A)}\right|}{\left|\vec{\Omega}_{LT}^{(E)}\right|} = \left(1 + \lambda_1 r + \lambda_1^2 r^2\right) e^{-\lambda_1 r} \tag{5.6}$$

$$\frac{\Omega_{G}^{(A)}}{\Omega_{G}^{(E)}} = \frac{\left|\vec{\Omega}_{G}^{(A)}\right|}{\left|\vec{\Omega}_{G}^{(E)}\right|} = \frac{1}{3}\left|(1 + \lambda_0 r)e^{-\lambda_0 r} - 8(1 + \lambda_1 r)e^{-\lambda_1 r}\right| \tag{5.7}$$

It is likely that the range of forces for the Yukawa type, in which additional intermediate-range forces could be added to the Newtonian component without being detected, would be 10 m $< \lambda^{-1} <$ 1 km where experimental data are poorest (Fujii, 1971; Long, 1974; Mikkelson and Newman, 1977). Taking λ_0^{-1} and λ_1^{-1} in Eqs. (5.6) and (5.7) to be 1 km, we obtain

$$\frac{\Omega_{LT}^{(A)}}{\Omega_{LT}^{(E)}} \sim 0, \quad \frac{\Omega_{G}^{(A)}}{\Omega_{G}^{(E)}} \sim 0$$

We see from above results that the precessions of an orbiting gyroscope circling the Earth predicted by higher-order gravitational theory are consistent with that in general relativity.

For the case of an orbiting gyroscope circling a spinning neutron star with radius $r_0 = 1$ km on an orbit with altitude $h = 9$ km, we have

$$\frac{\Omega_{LT}^{(A)}}{\Omega_{LT}^{(E)}} \sim 5 \times 10^{-3}, \quad \frac{\Omega_{G}^{(A)}}{\Omega_{G}^{(E)}} \sim 1.28 \times 10^{-3}$$

Thus the effects of the additional forces on the precession are much less than that of general relativity. If the ranges of additional forces were longer, or the radii of a neutron star and a gyroscope orbit are less, their effects might be comparable with (or larger than) the relativistic effect on the precessions of an orbiting gyroscope.

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